Exchange of quantum states between coupled oscillators

D. Portes Jr. and H. Rodrigues
Centro Federal de Educação Tecnológica do Rio de Janeiro
Departamento de Educação Superior-DEPES.

S. B. Duarte*
Centro Brasileiro de Pesquisas Físicas /CNPq,
Rua Dr. Xavier Sigaud 150, CEP 22.290-180,
Rio de Janeiro (RJ), Brazil.

B. Baseia
Instituto de Física, Universidade Federal de Goiás
PO-Box-131, 74.001-970, Goiania(GO), Brazil.

February 1, 2008

Abstract

Exchange of quantum states between two interacting harmonic oscillator along their evolution time is discussed. It is analyzed the conditions for such exchange starting from a generic initial state and demonstrating that the effect occurs exactly only for the particular states $C_0 \left|0 >+C_N \right|N >$, *which includes the interesting qubits components* $|0\rangle, |1\rangle$. *It is also determined the relation between the coupling constant and characteristic frequencies of the oscillators to have the complete exchange.*

1 Introduction

The engineering of quantum states of light fields and oscillators became an interesting topic in the last years, due to its applications in : (i) fundamentals of quantum mechanics (preparation of Schrodinger-cat states [1], their superposition [2] and measurement of their decoherence [3], etc.); (ii) determination of certain properties of a system (phase distribution P(θ) [4], Wigner [5] and Husimi [6] functions, etc.); (iii) proposals for practical applications (quantum lithography [7], quantum communication [8] - e.g., via hole-burning in Fock

*corresponding author, e-mail : sbd@cbpf.br

space [9] - quantum teleportation [10], etc). However, a difficult situation appears when one wants to prepare a state of a system offering hard access [11]. In this case the difficulty may be circumvented by coupling the system having hard access to a second system offering easy access, in which a desired state is prepared with subsequent transfer to the first one. The success of this operation depends on the model-Hamiltonian and on the initial state describing the whole system.

Although the problem of two interacting harmonic oscillators has been exhaustively studied in the literature, the discussion about exchange of nonclassical states between them **is** scarce. The coupled quantum oscillation problem was considered earlier in [12, 13, 14], where the authors of those papers were interested only in the energy of the system. Later on, in Ref [15] a full exchange between quantum two-mode harmonic oscillators was presented, however the issue was only concerned with the particular transfer of coherent states. In Ref. [16] we have studied the transfer of certain properties (statistics and squeezing) and in Ref. [17] we have studied the transfer of **the most** relevant part of the state of a sub-system to another, through the simultaneous transfer of the number and phase distributions, P_n and $P(\theta)$[1] [17]; the solutions were found numerically since the models were not exactly soluble.

In the present work we employ a distinct model-Hamiltonian, allowing us to treat the problem analytically permitting us to analyze the transfer of generic states. We show in which way one can get exact exchange of the states between two interacting sub-systems. *Exchange of states* means *simultaneous transfer of states in two opposite directions*; so, it is more significant than the transfer of states in one direction as studied in [17]. In the present case the transfer of a state from the "*easy-oscillator*" to the "*hard-oscillator*" is observed by simply monitoring the state of the easy-oscillator during the time evolution of the whole system. For brevity, hereafter the easy- and the hard-oscillator will be referred to as $\mathbf{O}_1$ and $\mathbf{O}_2$, respectively.

The Sect. II introduces the model-Hamiltonian allowing us to obtain the evolution operator for this coupled system. In the Sect. III we consider different types of initial states describing the entire system to study the mentioned effect between the $\mathbf{O}_1$ and the $\mathbf{O}_2$ (Sub-Sects. **(A)**, **(B)**,and **(C)**), including superpositions of states representing the qubits $|0\rangle$ and $|1\rangle$. The Sect. IV contains the comments and conclusion.

2 Model-Hamiltonian: evolution operator

We start from the Hamiltonian

$$\mathbf{H}/\hbar = \omega_1 \mathbf{a}_1^+ \mathbf{a}_1 + \omega_2 \mathbf{a}_2^+ \mathbf{a}_2 + \lambda \left(\mathbf{a}_1^+ \mathbf{a}_2 + \mathbf{a}_1 \mathbf{a}_2^+ \right) \ , \tag{1}$$

[1] Since the number and phase are canonically conjugate operators they are complementary, in the sense that simultaneous transfer of number and phase distributions, P_n and $P(\theta)$, concerns the transfer of the major part of the state describing a system.

where $\mathbf{a}_i^+(\mathbf{a}_i)$ stands for the raising (lowering) operator of the $i-th$ oscillator, $i=1,2;$ ω_i and λ are real parameters standing for the i-th oscillator frequency and coupling constant, respectively. The equations of motion for the operators $a_1(t)$ and $a_2(t)$ can be solved analytically,

$$\begin{aligned}
\mathbf{a}_1(t) &= \left(c^2 e^{-i\omega_1' t} + s^2 e^{-i\omega_2' t}\right)\mathbf{a}_1(0) + cs\left(e^{-i\omega_1' t} - e^{-i\omega_2' t}\right)\mathbf{a}_2(0), \qquad (2)\\
\mathbf{a}_2(t) &= \left(c^2 e^{-i\omega_2' t} + s^2 e^{-i\omega_1' t}\right)\mathbf{a}_2(0) + cs\left(e^{-i\omega_1' t} - e^{-i\omega_2' t}\right)\mathbf{a}_1(0),
\end{aligned}$$

where,

$$\begin{aligned}
\omega_1' &= \omega_1 + \lambda\frac{s}{c}\ , \qquad (3)\\
\omega_2' &= \omega_2 - \lambda\frac{s}{c}\ .
\end{aligned}$$

and

$$\begin{aligned}
s &= \left(\frac{1}{2} - \frac{x}{2\sqrt{x^2+1}}\right)^{1/2} \qquad (4)\\
c &= \left(\frac{1}{2} + \frac{x}{2\sqrt{x^2+1}}\right)^{1/2}
\end{aligned}$$

with

$$x = \frac{\omega_1 - \omega_2}{2\lambda}\ . \qquad (5)$$

The parameter s and c satisfy the condition $c^2+s^2=1$, they define the auxiliary operators

$$\begin{aligned}
\mathbf{a}_1' &= c\,\mathbf{a}_1 + s\,\mathbf{a}_2\ , \qquad (6)\\
\mathbf{a}_2' &= -s\,\mathbf{a}_1 + c\,\mathbf{a}_2\ ,
\end{aligned}$$

which decouple the above Hamiltonian. The following relations also hold:

$$\begin{aligned}
\omega_1' + \omega_2' &= \omega_1 + \omega_2\ , \qquad (7)\\
\omega_1' - \omega_2' &= \frac{\lambda}{cs}\ .
\end{aligned}$$

It is convenient for our purposes to find the time dependent state vector or density operator in the Schrodinger picture. One formal prescription is to work with Wigner representation of the state and obtain the time-dependent density operator from the Wigner function[19], for which the time evolution is easily obtained. However, it is a hard task to restore analytical or numerical values for the density matrix $\rho(t)$ in the Fock basis from the time dependent Wigner function. To overcome this difficult we will show that for the Hamiltonian given by Eq.(1) there is an analytical expression for the evolution operator $U(t)$, which defines the solution of the Schrodinger equation, allo**wing us** to get directly the matrix $\rho(t)$ in the Fock basis. This kind of approach was already used in Ref

[18], but only treating the system in the resonant case ($\omega_1 = \omega_2$). **I**n [18] the author studied the transfer of state starting from the particular one photon state. Our results permit one to obtain an analytical expression for the matrix element $U(t)$, for the Hamiltonian (1) not restricted to the resonant case and permitting easy application to a generic initial state. Consequently, the problem of transfer of states can be more comfortably discussed using the present results.

To obtain the **operator** $U(t)$, we define the **(auxiliary)** unitary operator $\mathbf{U}_s(t)$ which is associated to a rotation and decouples the Hamiltonian,

$$\mathbf{U}_s^{-1}\,\mathbf{a}_i\,\mathbf{U}_s = a_i' \ . \tag{8}$$

We have,

$$\mathbf{U}_s^{-1} = \mathbf{U}_{-s} \ , \tag{9}$$

in view of the reverse transformation

$$\begin{aligned} \mathbf{a}_1 &= c\,\mathbf{a}_1' - s\,\mathbf{a}_2' \ , \\ \mathbf{a}_2 &= s\,\mathbf{a}_1' + c\,\mathbf{a}_2' \ . \end{aligned} \tag{10}$$

We denote $\{|n_1, n_2\rangle_0\}$ as representing the Fock′s basis, eigenvectors of the (old) number operator $\mathbf{N}_i = \mathbf{a}_i^{+}\mathbf{a}_i$ **,** whereas $\{|n_1, n_2\rangle_s\}$ is the same for the (new) number operator $\mathbf{N}_i(s) = \mathbf{a}_i'^{+}\mathbf{a}_i'$. We have,

$$\begin{aligned} \mathbf{U}_s|n_1, n_2\rangle_s &= |n_1, n_2\rangle_0, \\ |n_1, n_2\rangle_s &= \mathbf{U}_{-s}|n_1, n_2\rangle_0. \end{aligned} \tag{11}$$

If we represent $\mathbf{U}_s$ in the Fock′s basis $\{|n_1, n_2\rangle_0\}$, we obtain

$$\begin{aligned} (\mathbf{U}_s)^{n_1,\,n_2}_{m_1,\,m_2} &= {}_0\langle n_1, n_2|\mathbf{U}_s|m_1, m_2\rangle_0 \\ &= {}_s\langle n_1, n_2|m_1, m_2\rangle_0. \end{aligned} \tag{12}$$

Next, to reconstruct the operator $\mathbf{U}_s$ in the Fock's basis, we start from

$${}_s\langle n_1, n_2|\,\mathbf{a}_1'|m_1, m_2\rangle_0 = {}_s\langle n_1, n_2|\,(c\,\mathbf{a}_1 + s\,\mathbf{a}_2)\,|m_1, m_2\rangle_0, \tag{13}$$

Since the operators $\mathbf{a}_i'$ act on the basis $\{|n_1, n_2\rangle_s\}$ whereas the $\mathbf{a}_i$ act on the basis $\{|n_1, n_2\rangle_0\}$, we get

$$\begin{aligned} \sqrt{n_1+1}\,{}_s\langle n_1+1, n_2|m_1, m_2\rangle_0 = c\sqrt{m_1}\ {}_s\langle n_1, n_2|m_1-1, m_2\rangle_0 \\ +s\sqrt{m_2}\ {}_s\langle n_1, n_2|m_1, m_2-1\rangle_0, \end{aligned} \tag{14}$$

which, **after** using the Eq.(12), leads to

$$(\mathbf{U}_s)^{n_1,\,n_2}_{m_1,\,m_2} = c\sqrt{\frac{m_1}{n_1}}\,(\mathbf{U}_s)^{n_1-1,n_2}_{m_1-1,m_2} + s\sqrt{\frac{m_2}{n_1}}\,(\mathbf{U}_s)^{n_1-1,n_2}_{m_1,m_2-1} \ , \tag{15}$$

and similarly, repeating the procedure for the operator $\mathbf{a}'_2$, we find

$$(\mathbf{U}_s)^{n_1,\,n_2}_{m_1,\,m_2} = -s\sqrt{\frac{m_1}{n_2}}\,(\mathbf{U}_s)^{n_1,n_2-1}_{m_1-1,m_2} + c\sqrt{\frac{m_2}{n_2}}\,(\mathbf{U}_s)^{n_1,n_2-1}_{m_1,m_2-1}\ . \tag{16}$$

Using the Eqs. (15), (16) plus the unitary condition $U_s^\dagger U_s = U_s U_s^\dagger = 1$ we obtain, after a lengthy calculation, the expression

$$\begin{aligned}(\mathbf{U}_s)^{n_1,\,n_2}_{m_1,\,m_2} \quad &= \quad \delta_{n_1+n_2,\,m_1+m_2}\sqrt{\frac{n_1!n_2!}{m_1!m_2!}}\,(-1)^{n_2}\,c^{m_1-n_2}\,s^{m_2+n_2} \\ &\times \sum_{k=\max(0,m_2-n_1)}^{\min(n_2,m_2)} (-1)^{-k}\left(\frac{s}{c}\right)^{-2k}\binom{m_1}{n_2-k}\binom{m_2}{k}\,,\end{aligned} \tag{17}$$

and

$$(\mathbf{U}_{-s})^{n_1,\,n_2}_{m_1,\,m_2} = (-1)^{m_2-n_2}\,(\mathbf{U}_s)^{n_1,\,n_2}_{m_1,\,m_2}\ . \tag{18}$$

The time evolution operator $U(t)$ may be written in the basis $\{|n_1,n_2\rangle_s\}$ as

$$\mathbf{U(t)} = \sum_{k_1,k_2} |k_1,k_2\rangle_s\ e^{-i(k_1\omega'_1+\,k_2\omega'_2)t}\ {}_s\langle k_1,k_2|\ , \tag{19}$$

for H is diagonal in this basis. Finally from the Eqs.(12) and (19) we obtain the expression

$$\mathbf{U(t)}^{n_1,\,n_2}_{m_1,\,m_2} = \sum_{k_1,k_2} e^{-i(k_1\omega'_1+\,k_2\omega'_2)t}\,(\mathbf{U}_{-s})^{n_1,\,n_2}_{k_1,\,k_2}\,(\mathbf{U}_{-s})^{m_1,\,m_2}_{k_1,\,k_2}\ , \tag{20}$$

restricted to $n_1+n_2 = k_1+k_2 = m_1+m_2$, whereas $U^{n_1,\,n_2}_{m_1,\,m_2} = 0$ otherwise.

The evolution operator obtained in Eq.(20) allows us to study the time evolution of the whole state describing our bipartite system composed by coupled oscillators, represented by the Hamiltonian in the Eq.(1). In the next section we will study the exchange of states between these oscillators and, as a natural assumption, we will suppose the $\mathbf{O}_2$ initially in its ground state $|0\rangle$. The $\mathbf{O}_1$ is assumed to be previously prepared in various initial states, firstly starting from an arbitrary state $|\phi\rangle$.

3 Exchange of generic state

Let us consider that the whole (bipartite) system is initially in the state

$$|\Psi(0)\rangle\ =\ |\phi\rangle\otimes|0\rangle\,, \tag{21}$$

whose components in the Fock's basis are given by,

$$|\Psi(0)\rangle\ = \sum_n C^{n,\,0}(0)|n,0\rangle\,, \tag{22}$$

since $C^{n_1,\, n_2}(0) = 0$ for $n_2 \neq 0$. In the Schrodinger representation, the coefficients $C^{n_1,n_2}(t)$ are obtained from $C^{n_1,n_2}(t) = \langle n_1, n_2|\mathbf{U(t)}|\Psi(0)\rangle$, which, using Eq. (22) and the constraint $n_1 + n_2 = n$, results in the form

$$C^{n_1,n_2}(t) = C^{n_1+n_2,0}(0)\, \mathbf{U(t)}^{n_1,n_2}_{n_1+n_2,0} \,. \tag{23}$$

In particular, we have that

$$C^{n,0}(t) = C^{n,0}(0)\, \mathbf{U(t)}^{n,0}_{n,0} \,, \tag{24}$$

and

$$C^{0,n}(t) = C^{n,0}(0)\, \mathbf{U(t)}^{0,n}_{n,0} \,. \tag{25}$$

The exchange of states between the oscillators will occur after an instant τ,when $C^{0,n}(\tau) = C^{n,0}(0)$ and

$$|\Psi(\tau)\rangle = \sum_n C^{0,\, n}(\tau)|0, n\rangle \,, \tag{26}$$

or, $|\Psi(\tau)\rangle = |0\rangle \otimes |\phi\rangle$. This shows that exchange of states allows us to verify the transfer of states to the $\mathbf{O}_2$ by monitoring the time evolution of the $\mathbf{O}_1$.

From the Eqs. (17) and (18) we have,

$$(\mathbf{U}_s)^{n,0}_{n-l,l} = \sqrt{\frac{n!}{(n-l)!l!}} c^{n-l}\, s^l \,, \tag{27}$$

and

$$(\mathbf{U}_s)^{0,n}_{n-l,l} = \sqrt{\frac{n!}{(n-l)!l!}}\, (-1)^{n-l}\, c^l\, s^{n-l} \,. \tag{28}$$

The substitution of the Eqs. (27) and (28) in the Eq. (20) results

$$\mathbf{U(t)}^{0,n}_{n,0} = (-1)^n \sum_{l=0}^{n} \frac{n!}{(n-l)!l!}\, (-1)^{n-l}\, c^n s^n e^{-i\,(n-l)\ \omega_1'\ t}\, e^{-i\,l\ \omega_2'\ t} \,. \tag{29}$$

where we recognize the Newton's binomial expression,

$$\mathbf{U(t)}^{0,n}_{n,0} = (-1)^n\, c^n s^n \left(e^{-i\,\omega_2'\ t} - e^{-i\,\omega_1'\ t}\right)^n \tag{30}$$

or, replacing the auxiliary parameters ω_1', ω_2' by ω_1, ω_2 and λ (cf. Eq. (7)),

$$\mathbf{U(t)}^{0,n}_{n,0} = e^{-i\frac{\omega_1+\omega_2}{2}\, n\, t} \left(-2\, i\, s\, c \sin(\frac{\lambda}{2\, cs}\, t)\right)^n \,, \tag{31}$$

and, consequently,

$$C^{0,n}(t) = C^{n,0}(0)\, e^{-i\frac{\omega_1+\omega_2}{2}\, n\, t} \left(-2\, i\, s\, c \sin(\frac{\lambda}{2\, cs}\, t)\right)^n \,. \tag{32}$$

In a similar way we get,

$$C^{n,0}(t) = C^{n,0}(0)\, e^{-i\, \frac{\omega_1+\omega_2}{2}\, n\, t} \left(c^2 e^{-i\, \frac{\lambda}{2cs}\; t} + s^2 e^{i\, \frac{\lambda}{2cs}\; t} \right)^n \; . \tag{33}$$

From Eq. (32) we see that a partial exchange of states will occur when $\lambda t/sc = (2k+1)\,\pi,$ i.e., in the time intervals $\tau_k = (sc/\lambda)\ (2k+1)\,\pi$. The effect attains the highest efficiency when the product sc is maximum, i.e., when $s = c = 1/\sqrt{2}$ and $\tau_k = (k+1/2)\,\pi/\lambda$. According to the Eq. (4) this implies $x = 0$ and the resonance condition $\omega_1 = \omega_2 = \omega$ (cf. Eq. (5)),

$$C^{0,n}(\tau_k) = (-i)^n\, C^{n,0}(0)\, e^{-i\,\omega\, n\, \tau_k}. \tag{34}$$

However, we note that even at resonance we obtain no exchange of states, due to the presence of the phase factor $\exp\left[-i\, \left(\omega\tau_k + \frac{\pi}{2}\right)\, n\right]$ affecting the coefficients of the state describing both oscillators in the Fock's representation. In this general case we obtain $\left|C^{0,n}(\tau_k)\right| = \left|C^{n,0}(0)\right|$, which means exchange of statistics between the two oscillators. This can also be seen comparing both reduced density matrix, $\rho^{(2)}_{m_1,\, m_2}(\tau_k)$ and $\rho^{(1)}_{m_1,\, m_2}(0)$, in the Fock's representation,

$$\rho^{(2)}_{m_1,\, m_2}(\tau_k) = e^{-i\,\left(\omega\tau_k + \frac{\pi}{2}\right)\,(m_1 - m_2)}\ \rho^{(1)}_{m_1,\, m_2}(0)\ , \tag{35}$$

which exhibits the distinction between their off-diagonal elements. As well known, while the state of a system offers its complete description, the same is not true for the statistics, which contains only partial informations of the system.

3.1 The complete exchange of state

It is shown in the last section that it is not possible to have a complete exchange of states for a generic initial state because the phases are not transferred (see Eq.35). Here we show that when the state of oscillator O_1 is given by the superposition $C_0|0\rangle + C_N|N\rangle$ whereas O_2 is in the vacuum state, complete exchange of states occurs. Note that this state includes in particular the important case $C_0|0\rangle + C_1|1\rangle$ using the qubits $|0\rangle$, $|1\rangle$ having potential applications in quantum communication [20] and in quantum computation [21]. It was shown that this state exhibits squeezed fluctuations [22].

Next, let us consider the whole system initially in the superposed state

$$|\Psi(0)\rangle \ = \ C^{0,0}(0)|0,0\rangle + C^{N,0}(0)|N,0\rangle\,. \tag{36}$$

In this case we verify perfect exchange of states between the oscillators for a convenient choice of the parameters involved. Assuming the resonance condition in the Eq.(32) we have, for $C^{0,0}(t) = C^{0,0}(0)$,

$$C^{0,N}(t) = C^{N,0}(0)\, e^{-i\,(\omega\ t + \pi/2)\, N}\, \sin^N(\lambda\, t)\ . \tag{37}$$

Partial exchange of states will occur when $t = \tau_0 = \pi/(2\lambda)$, which results in

$$C^{0,N}(\tau_k) = C^{N,0}(0)\, e^{-i\,\pi/2(\omega/\lambda+1)\,N} \ , \tag{38}$$

whose meaning is the exchange of statistics. The exchange of states becomes complete (exact) when $C^{0,N}(\tau_k) = C^{N,0}(0)$, namely, when

$$\frac{\omega}{\lambda} = \frac{4m - N}{N} \ \ , \tag{39}$$

with m integers. Taking $m = 1$ and ω in the microwave domain ($\omega \sim 10^9 Hz$) the time spent to transfer the state $C_0|0> + C_1|1>$ from the $\mathbf{O}_1$ to the $\mathbf{O}_2$ results $\tau_0 = \pi/(2\lambda) \sim 10^{-9}$s, since $\lambda = \omega/3$ (cf. Eq.(39)), which is smaller than the typical decoherence time for such systems ($\tau_d \sim 10^{-3}s$), as it should.

Note that the previous initial state $C_0|0\rangle + C_N|N\rangle$ describing the O_1 includes the Fock states $|N\rangle$, obtained from $C_0 = 0$ and $C_N = 1$. In this case exact exchange of states no longer requires the Eq. (39). The reason comes from the phase factor appearing in the Eq. (39), now becoming a global phase with no physical relevance. In this case the exchange of states is exact for any instant $t_k = \tau_0 + 2\pi k/\lambda$.

4 Comments and Conclusion

An analytical procedure applied to a convenient model-Hamiltonian describing two coupled oscillators allows us to get the exact evolution operator for the entire system (Sect. II). This approach, through the use of distinct initial states and parameters (Sub-Sects. **(A), (B)** of Sect. III), makes easy the study of exchange of states between such sub-systems. In all cases we have shown that the fidelity of the process is maximum when the resonance condition, $\omega_1 = \omega_2$, is attained. Assuming the $\mathbf{O}_2$ always in the vacuum state we find, sub-Section by sub-Section, that: (**A**) partial exchange of states is achieved when the initial state of the $\mathbf{O}_1$ is arbitrary, for the time intervals $t = \tau_k = (k + 1/2)\pi/\lambda$; the efficiency of partial exchange is maximum when the product sc is maximum ($sc = 1/2$); however, while the occurrence of exchange of states is partial, exchange of *statistics* is obtained exactly, as shown in the Eqs. (34), (35); **(B)** exact exchange of states occurs when the $\mathbf{O}_1$ starts from the initial superposed state $C_0|0\rangle + C_N|N\rangle$, in the time intervals $t_k = \tau_0 + 2\pi k/\lambda$, with the requirement in Eq. (39). If the Eq.(39) is not obeyed, exchange of states will occur at the same time intervals, but now the effect is only partial; Exact exchange of states is also found in the particular case of (B), setting $C_0 = 0$ and $C_N = 1$, which means the $\mathbf{O}_1$ starting from a Fock state $|N\rangle$. In this case the exchange of states occurs exactly at the same time intervals found in (B), no matter the Eq. (39) is obeyed or not.

As final remarks we mention that exchange of states and its efficiency could be investigated for other model-Hamiltonians and, as explained before, the effect goes beyond those studied in [16] and [17]. To our knowledge, exchange of

states in coupled systems and even exchange of certain properties, are subjects receiving little attention in the literature [23] - with the remarkable exception of quantum teleportation [21], an effect having a very distinct nature (requiring the presence of quantum channels and entangled states), which occurs in the absence of coupling between the two sub-systems. In the context of teleportation, exchange of states appears with the name "identity interchange" [24] and "two-way teleportation" [25].

4.1 Acknowledgements

The authors thank the CNPq (SBD, BB) and FAPERJ (DPJ) for the partial supports.

4.2 References

References

[1] B.Yurke, D. Stoler, Phys. Rev. Lett. **57** (1986) 13.

[2] L. Davidovich et al., Phys. Rev. Lett. **71** (1993) 2360.

[3] M. Brune et al., Phys. Rev. Lett. **77** (1996) 4887; D.M. Meekhof et al., Phys. Rev. Lett. **76** (1996) 1796.

[4] D.T. Pegg, S. M. Barnett, Phys. Rev. Lett. **76** (1996) 4148.

[5] L.G. Lutterbach and L. Davidovich, Phys. Rev. Lett. **78** (1997) 2547.

[6] M. H.Y. Moussa, B. Baseia, Phys. Lett. A **238** (1998) 223.

[7] G. Bjork, L.L. Sanchez-Soto, J. D. Soderholm, Phys. Rev. Lett. **86** (2001) 4516.

[8] See, e.g., S.L. Braustein, P. van Loock, Rev. Mod. Phys. **77** (2005) 513.

[9] B. Baseia, J.M.C. Malbouisson, Chinese Phys. Lett. **18** ,1467 (2001); Phys. Lett. A **290** (2001) 234; A.T. Avelar, B. Baseia, Opt. Commun. **239** (2004) 281; Phys. Rev. A **72** (2005) 67508; B. Escher et al., Phys. Rev. A **70** (2004) 025801.

[10] B. Julsgaard et al., Nature, **413** (2001) 400, and references therein.

[11] F. Dietrich et al., Phys. Rev. Lett. **62** (1989) 403; D.J. Heizein et al., Phys. Rev. Lett. **66** (1991) 2080.

[12] J. Tucker and D. F. Walls, Ann. Phys. (N.Y.) **52**, 1 (1969).

[13] E.Y.C. Lu, Phys. Rev. A 8, 1053 (1973).

[14] M.S. Abdalla, J. Phys. A: Math. Gen. 29, 1997 (1996).

[15] Marcos C de Oliveira et al, Journal Optics B **1** (1999) 610.

[16] H. Rodrigues et al., Physica A **311** (2002)188.

[17] D. Portes Jr., et al., Physica A **329** (2003) 391.

[18] Lee E. Estes, Thomas H. Keil, and Lorenzo M. Narducci, Physical Review **175,1** (1968) 286.

[19] B. R. Mollow, Physical Review **162,5** (1967) 1256.

[20] S. J. van Enk, J. I. Cirac, P. Zoller, Phys. Rev. Lett. **78** (1997) 4293.

[21] P. W. Shor, Phys. Rev. A **52** (1995) R2493.

[22] K. Wodkiewicz et al, Phys. Rev. A **35**, (1987) 2567.

[23] A. S. M. de Castro, V.V. Dodonov , J. Opt. B: Quantum Semiclass. Opt. **4** (2002) 191.

[24] M. H. Y. Moussa, Phys. Rev. A **55**, (1997) R3287.

[25] L. Vaidman, N. Yoran, Phys. Rev. A **59** (1999) 116.